1° Curvature of plane curves

1.1. Computation from $k := \frac{d\alpha}{ds}$

For a curve $\alpha : \begin{cases} x = x(t) \\ y = y(t) \end{cases}$, we find

$$\alpha = \arctan \frac{y'}{x'}, \qquad s = \int_0^t \sqrt{x'^2 + y'^2}\, d\tau$$

Hence

$$d\alpha = \frac{y''x' - x''y'}{x'^2 + y'^2}\, dt, \qquad ds = \sqrt{x'^2 + y'^2}\, dt,$$

$$\left|\frac{d\alpha}{ds}\right| = |x'y'' - x''y'| \Big/ (x'^2 + y'^2)^{3/2}$$

1.2. Another definition

We introduce arc-length parameter for $r : \begin{cases} x = x(t), \\ y = y(t). \end{cases}$

Write $r(t) := (x(t), y(t))$. (Vector Valued)

In general, $\left|\frac{dr}{dt}\right| \neq 1$ While we can take

$$s(t) := \int_0^t \left|\frac{dr}{dt}(\tau)\right| d\tau$$

We calculate $\left|\frac{dr}{ds}\right|$ as follow:

$$\left|\frac{dr}{dt}\right| = \left|\frac{dr}{ds}\right| \left|\frac{ds}{dt}\right| = \left|\frac{dr}{ds}\right| \left|\frac{dr}{dt}\right| \Rightarrow \left|\frac{dr}{ds}\right| \equiv 1.$$

Def. For a curve $r = r(s)$ parametrized by arc-length, the curve k is defined by $k := r''(s)$.

Next we shall calculate κ for $r(t) = (x(t), y(t)) \subset \mathbb{R}^2$.

First, we change t for arc-length parameter s. Then

$$r(t) = r(s(t)) = (x(s(t)), y(s(t)))$$

$$\frac{dr}{ds} = \frac{dr}{dt}\frac{dt}{ds} = (\dot{x}, \dot{y}) \Big/ \sqrt{\dot{x}^2 + \dot{y}^2}$$

$$\kappa = \frac{d^2 r}{ds^2} = \frac{d}{dt}\left(\frac{(\dot{x}, \dot{y})}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right)\frac{dt}{ds} = \frac{\ddot{x}\dot{y} - \dot{x}\ddot{y}}{(\dot{x}^2 + \dot{y}^2)^2}\,(\dot{y}, -\dot{x})$$

(The Newton point diff. means to diff w.r.t. t).

Rmk. $|\kappa| = k$.

2°. Mean Value Principle (MVP).

Mean Thms: Roll MVP, Lagrange MVP, Cauchy MVP

Rmk. MVP's are useful tools to solve problems about existence

First, consider the following problems.

Q2.1 Suppose $f \in C[1,2]$ is differentiable on $(1,2)$, and $f(2)=2$. $f(1)=\frac{1}{2}$. Prove that:

$$\exists \xi \in (1,2) \quad \text{s.t.} \quad f'(\xi) = \frac{2f(\xi)}{\xi}$$

Construction: $F(x) = \frac{f(x)}{x^2}$

Q2.2. Suppose $f \in C[0,\pi]$ and f is twice differentiable on $(0,\pi)$. Assume $f(0)=f(\pi)=0$. Prove that:

$$\exists \xi \in (0,\pi) \quad \text{s.t.} \quad f(\xi) + f''(\xi) = 0.$$

Construction: $F(x) = f'(x)\cos x - f(x)\sin x$

It's meaningful to ask how comes the constructions. Here we actually solve ODE's to give those (eg. For Q2.1. we solve $\frac{dy}{dx} = \frac{2y}{x}$ to find $\frac{y}{x^2} \equiv \text{const}$)

Rmk. Such a method is machinacal; however it is rather limited.

Take a look at following more delicate construction.

Q2.3. Suppose $f \in C[0,1]$ is differentiable on $(0,1)$ with $f(0)=0$, $f(1)=1$.

Prove that for $r_1, \dots, r_n > 0$ with $\sum_{i=1}^n r_i = 1$, there exists $x_1, \dots, x_n \in [0,1]$ which are distinct from each other s.t.

$$\sum_{i=1}^n \frac{r_i}{f'(x_i)} = 1$$

Sketch of proof

For $i=1,\dots,n$, $\exists\ u_i \in (u_{i-1}, 1]$ s.t. $f(u_i) = \sum_{j=1}^{i} r_j$, with $\begin{cases} u_n := 1, \\ u_0 := 0. \end{cases}$

By Lagrange MVP, we find $\exists\ x_i \in (u_{i-1}, u_i)$ s.t.

$$r_i = f(u_i) - f(u_{i-1}) = f'(x_i)(u_i - u_{i-1})$$

Hence $\sum_{i=1}^n \frac{r_i}{f'(x_i)} = \sum_{i=1}^n (u_i - u_{i-1}) = u_n - u_0 = 1$ $\square$

3° Taylor's Formula.

If $f \in C^n(a,b)$, $x_0 \in (a,b)$ then we write

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2!} f^{(2)}(x_0)(x-x_0)^2 + \cdots + \frac{1}{n!} f^{(n)}(x_0)(x-x_0)^n + R_n(x)$$

Peano : $R_n(x) = o(x-x_0)^n$

Lagrange: If further assume $f \in C^{n+1}(a,b)$, then $\exists \xi(x)$ s.t.

$\xi(x) \in (x_0, x)$ or (x, x_0), and $R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi(x))(x-x_0)^{n+1}$

4° Some problems on calculation of limits

4.1. $\lim\limits_{x \to +\infty} (x^{\frac{1}{x}} - 1)^{\frac{1}{\ln x}}$

$$= \lim_{x\to+\infty} \exp\left\{\frac{\ln(x^{1/x}-1)}{\ln x}\right\} = \exp\left\{\lim_{t\to 0^+} \frac{\ln\left(\frac{1}{t^t}-1\right)}{-\ln t}\right\}$$

$$= \exp\left\{\lim_{t\to 0^+} \frac{-\ln(1-e^{t\ln t}) + t\ln t}{\ln t}\right\} = \exp\left\{\lim_{t\to 0^+} \frac{-\ln(-t\ln t - O(t^2\ln^2 t))}{\ln t}\right\}$$

$$= \exp\left\{-1 - \lim_{s\to-\infty} \frac{\ln(-s - O(s^2 e^s))}{s}\right\} = \frac{1}{e}$$

4.2. $\lim\limits_{x\to\infty} \frac{x}{1+e^{-x}} - \ln(1+e^x)$

$$= \lim_{x\to\infty} x(1-e^{-x}+O(e^{-2x})) - (e^{-x} - O(e^{-2x})) - x = 0$$