数学分析 (II)

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1 不定积分

1.1.1. 求不定积分 $\int \frac{mx+n}{x^2+px+q}dx$

ans : 将分母进行配方 $x^2+px+q=(x+\frac{1}{2}p)^2+q-\frac{1}{4}p^2$

再设

$$t=x+\frac{1}{2}p$$

则

$$x=t-\frac{1}{2}p, dx=dt$$

并令

$$q-\frac{1}{4}p^2=\pm a^2$$

（等式右边符号取决于 q 和 $\frac{1}{4}p^2$ 的大小关系）

$$A=m, B=n-\frac{1}{2}mp$$

则

$$mx+n=At+B$$

于是我们可以把积分写成 $\int \frac{mx+n}{x^2+px+q}dx=\int \frac{At+B}{t^2\pm a^2}dt=A\int \frac{tdt}{t^2\pm a^2}+B\int \frac{dt}{t^2\pm a^2}$

$$A\int \frac{tdt}{t^2\pm a^2}=\frac{A}{2}\int \frac{d(t^2\pm a^2)}{t^2\pm a^2}=\frac{A}{2}\ln|t^2\pm a^2|+C$$

$$B\int \frac{dt}{t^2+a^2}=\frac{B}{a}\arctan\frac{t}{a}+C$$

$$B\int \frac{dt}{t^2-a^2}=\frac{B}{2a}\ln|\frac{t-a}{t+a}|+C$$

$$\int\frac{mx+n}{x^2+px+q}dx=\begin{cases}\frac{m}{2}\ln|x^2+px+q|+\frac{2n-mp}{\sqrt{4q-p^2}}arctan\frac{2x+p}{\sqrt{4q-p^2}}+C, & q>\frac{p^2}{4}\\ m\ln|x-\frac{1}{2}p|+\frac{\frac{1}{2}mp-n}{t-\frac{1}{2}p}+C, & q=\frac{p^2}{4}\\ \frac{m}{2}\ln|x^2+px+q|+\frac{2n-mp}{2\sqrt{p^2-4q}}\ln|\frac{2x+p-\sqrt{p^2-4q}}{2x+p-\sqrt{p^2+4q}}|+C, & q<\frac{p^2}{4}\end{cases}$$

$(i)\int\frac{dx}{(x-a)^k}$

$ans:$

$$\int\frac{dx}{(x-a)^k}=\begin{cases}\ln|x-a|+C, & k=1\\ \frac{1}{(1-k)(x-a)^{k-1}}+C & k>1\end{cases}$$

$(ii)\int\frac{Lx+M}{(x^2+px+q)^k}dx(p^2<4q)$

$ans:$ 令 $t=x+\frac{p}{2}$，原式便化为

$$\begin{aligned}\int\frac{Lx+M}{(x^2+px+q)^k}dx&=\int\frac{Lt+N}{(t^2+r^2)^k}dt\\&=L\int\frac{t}{(t^2+r^2)^k}dt+N\int\frac{dt}{(t^2+r^2)^k}\end{aligned}$$

其中，$r^2=q-\frac{1}{4}p^2,N=M-\frac{1}{2}pL$

当 $k=1$ 时，右边两个不定积分分别为

$$\int\frac{t}{(t^2+r^2)}dt=\frac{1}{2}\ln\left(t^2+r^2\right)+C$$
$$\int\frac{dt}{(t^2+r^2)}=\frac{1}{r}\arctan\frac{t}{r}+C$$

当 $k\geq 2$ 时，右边第一个不定积分为

$$\int\frac{t}{(t^2+r^2)^k}dt=\frac{1}{2(1-k)(t^2+r^2)^{k-1}}+C$$

对于第二个不定积分，记

$$I_k=\int\frac{dt}{(t^2+r^2)^k}$$

利用分部积分推导出递推公式

$$\begin{aligned} I_k &= \frac{1}{r^2}\int\frac{(t^2+r^2)-t^2}{(t^2+k^2)^k}dt \\ &= \frac{1}{r^2}I_{k-1}-\frac{1}{r^2}\int\frac{t^2}{(t^2+k^2)^k}dt \\ &= \frac{1}{r^2}I_{k-1}+\frac{1}{2r^2(k-1)}\int td(\frac{1}{(t^2+r^2)^{k-1}}) \\ &= \frac{1}{r^2}I_{k-1}+\frac{1}{2r^2(k-1)}[\frac{t}{(t^2+r^2)^{k-1}}-I_{k-1}] \end{aligned}$$

整理得

$$I_k=\frac{t}{2r^2(k-1)(t^2+r^2)^{k-1}}+\frac{2k-3}{2r^2(k-1)}I_{k-1}$$

反复递推，最归为计算 I_1。

由此我们可以解决很多有理代数分式的积分！

1.2.1. 求不定积分 $\int(2x^3+3x^2+4x+5)e^x dx$

ans：令

$$u=2x^3+3x^2+4x+5, dv=e^x dx, v=\int e^x dx=e^x$$

则

$$\begin{aligned} &u^{'}=6x^2+6x+4, && u^{''}=12x+6, && u^{'''}=12 \\ &v^{'}=e^x, && v^{''}=e^x, && v^{'''}=e^x \end{aligned}$$

于是

$$\begin{aligned} \int(2x^3+3x^2+4x+5)e^x dx &= \int uv^{(n+1)}dx \\ &= uv^{(n-1)}-u^{'}v^{(n-1)}+u^{''}v^{(n-2)}-u^{'''}v^{(n-3)} \\ &= (2x^3+3x^2+4x+5)e^x-(6x^2+6x+4)e^x+(12x+6)e^x-12e^x+C \\ &= (2x^3-3x^2+10x-5)e^x+C \end{aligned}$$

1.2.2. 求不定积分 $\int\cos x(x^3+2x^2+3x+4)dx$

ans：令

$$u = x^3 + 2x^2 + 3x + 4, dv = \cos x dx, v = \int \cos x dx$$

则

$$\begin{aligned} & & v^{(n+1)} &= \cos x \\ u &= x^3 + 2x^2 + 3x + 4 & v^{(n)} &= \sin x \\ u^{'} &= 3x^2 + 4x + 3 & v^{(n-1)} &= -\cos x \\ u^{''} &= 6x + 4 & v^{(n-2)} &= -\sin x \\ u^{'''} &= 6 & v^{(n-3)} &= \cos x \\ u^{(4)} &= 0 & v^{(n-4)} &= \sin x \end{aligned}$$

这样可以得到

$$\begin{aligned} \int (x^3 + 2x^2 + 3x + 4) \cos x dx &= \int uv^{(n+1)} dx \\ &= uv^{(n-1)} - u^{'} v^{(n-1)} + u^{''} v^{(n-2)} - u^{'''} v^{(n-3)} \\ &= (x^3 + 2x^2 + 3x + 4) \sin x - (3x^2 + 4x + 3)(-\cos x) \\ &\quad + (6x + 4)(-\sin x) - 6 \cos x + C \\ &= (x^3 + 2x^2 - 3x) \sin x + (3x^2 + 4x - 3) \cos x + C \end{aligned}$$

思考题（1）

$(i) \int P(x) e^{ax} dx$

ans：令 $v^{(n+1)} = e^{ax}$，则

$$v^{(n)} = \frac{e^{ax}}{a}, v^{(n-1)} = \frac{e^{ax}}{a^2}, v^{(n-2)} = \frac{e^{ax}}{a^3}, \cdots$$

并假设 $P(x)$ 是 x 的 n 次多项式，那么积分为

$$\int P(x) e^{ax} dx = e^{ax} (P \cdot \frac{1}{a} - P^{'} \cdot \frac{1}{a^2} + P^{''} \cdot \frac{1}{a^3} - P^{'''} \cdot \frac{1}{a^4} + \cdots) + C$$

$(ii) \int P(x) \sin bx dx$

ans：令 $v^{(n+1)} = \sin bx$, 则

$$v^{(n)} = -\frac{\cos bx}{b}, v^{(n-1)} = -\frac{\sin bx}{b^2}, v^{(n-2)} = \frac{\cos bx}{b^3}, \cdots$$

仍假定 $P(x)$ 是 x 的 n 次多项式，因此有

$$\int P(x)\sin bxdx = \sin bx(P^{'}\cdot\frac{1}{b^2} - P^{'''}\cdot\frac{1}{b^4}+\cdots) - \cos bx(P\cdot\frac{1}{b} - P^{''}\cdot\frac{1}{b^3}+\cdots)+C$$

$(iii)\int P(x)\cos bxdx$

ans : 令 $v^{(n+1)} = \cos bx$，则

$$v^{(n)} = \frac{\sin bx}{b}, v^{(n-1)} = -\frac{\cos bx}{b^2}, v^{(n-2)} = \frac{\sin bx}{b^3}, \cdots$$

这里，仍假定 $P(x)$ 是 x 的 n 次多项式，那么积分为

$$\int P(x)\cos bxdx = \sin bx(P\cdot\frac{1}{b} - P^{''}\cdot\frac{1}{b^3}+\cdots) + \cos bx(P^{'}\cdot\frac{1}{b^2} - P^{'''}\cdot\frac{1}{b^4}+\cdots)+C$$

$$\int x^k(\ln x)^n dx = \frac{1}{k+1}x^{k+1}(\ln x)^n - \frac{n}{k+1}\int x^k(\ln x)^{n-1}dx$$

$$\int e^{ax}\cos bxdx, \int e^{ax}\sin bxdx$$

设

$$u = \cos bx, dv = e^{ax}dx$$

$$du = -b\sin bx, v = \frac{1}{a}e^{ax}$$

那么

$$\begin{aligned}\int e^{ax}\cos bxdx &= \frac{1}{a}\cos bx - \int\frac{1}{a}e^{ax}(-b\sin bx)\\ &= \frac{1}{a}\cos bx + \frac{b}{a}\int e^{ax}\sin bxdx\\ \int e^{ax}\sin bxdx &= \frac{1}{a}\sin bx - \frac{b}{a}\int e^{ax}\cos bxdx\end{aligned}$$

$$\begin{cases}\int e^{ax}\cos bxdx = e^{ax}\frac{a\cos bx+b\sin bx}{a^2+b^2}+C\\ \int e^{ax}\sin bxdx = e^{ax}\frac{a\sin bx-b\cos bx}{a^2+b^2}+C\end{cases}$$

该方法也称为“配对积分法”

思考题 (2)

$(i)\int x^n e^{ax}\cos bxdx$

$(ii)\int x^n e^{ax}\sin bxdx$

$ans:$ 令

$$u = x^n, dv = e^{ax}\cos bx dx, du = nx^{n-1}dx,$$
$$v = e^{ax}\frac{a\cos bx + b\sin bx}{a^2+b^2}$$

$$\begin{aligned}\int x^n e^{ax}\cos bx dx &= x^n e^{ax}\frac{a\cos bx + b\sin bx}{a^2+b^2} - \int e^{ax}\frac{a\cos bx + b\sin bx}{a^2+b^2}\cdot nx^{n-1}dx \\ &= x^n e^{ax}\frac{a\cos bx + b\sin bx}{a^2+b^2} - \frac{na}{a^2+b^2}\int x^{n-1}e^{ax}\cos bx dx - \frac{nb}{a^2+b^2}\int x^{n-1}e^{ax}\sin bx dx \\ \int x^n e^{ax}\sin bx dx &= x^n e^{ax}\frac{a\sin bx - b\cos bx}{a^2+b^2} - \int e^{ax}\frac{a\sin bx - b\cos bx}{a^2+b^2}\cdot nx^{n-1}dx \\ &= x^n e^{ax}\frac{a\sin bx - b\cos bx}{a^2+b^2} - \frac{na}{a^2+b^2}\int x^{n-1}e^{ax}\sin x dx + \frac{nb}{a^2+b^2}\int x^{n-1}e^{ax}\cos bx dx\end{aligned}$$

1.3.1. 求积分 $\int \frac{a+x}{\sqrt{3a+2x}}dx$

$ans:$

$$\begin{aligned}\int \frac{a+x}{\sqrt{3a+2x}}dx &= \int \frac{ax+x^2}{\sqrt{3ax^2+2x^3}}dx \\ &= \frac{1}{6}\int \frac{d(3ax^2+2x^3)}{\sqrt{3ax^2+2x^3}} \\ &= \frac{1}{3}\sqrt{3ax^2+2x^3}+C \\ &= \frac{x}{3}\sqrt{3a+2x}+C\end{aligned}$$

1.3.2. 求积分 $\int \sqrt{ax+x^2}(3ax^3+4x^4)dx$

$ans:$

$$\begin{aligned}\int \sqrt{ax+x^2}(3ax^3+4x^4)dx &= \int \sqrt{ax^3+x^4}(3ax^2+4x^3)dx \\ &= \int \sqrt{ax^3+x^4}d(ax^3+x^4) \\ &= \frac{2}{3}(ax^3+x^4)\sqrt{ax^3+x^4}+C\end{aligned}$$

1.3.3. 求积分 $\int \frac{dx}{\sqrt{x}(1+\sqrt[4]{x})^3}$

ans : 设 $\sqrt[4]{x} = t$，则 $x = t^4, dx = 4t^3dt$, 代入积分式子

$$\int \frac{dx}{\sqrt{x}(1+\sqrt[4]{x})^3} = 4\int \frac{t^3dt}{t^2(1+t)^3} = 4\int \frac{tdt}{(1+t)^3} = 4\int [\frac{1}{(1+t)^2} - \frac{1}{(1+t)^3}]dt$$
$$= -\frac{4}{1+t} + \frac{2}{(1+t)^2} + C$$
$$= \frac{2}{(1+\sqrt[4]{x})^2} - \frac{4}{1+\sqrt[4]{x}} + C$$

1.3.4. 求积分 $\int \frac{xdx}{\sqrt{1+\sqrt[3]{x^2}}}$

ans : 令 $\sqrt{1+\sqrt[3]{x^2}} = y$，则

$$1+\sqrt[3]{x^2} = y^2, \sqrt[3]{x^2} = y^2-1, x = (y^2-1)^{\frac{3}{2}}, dx = 3y(y^2-1)^{\frac{1}{2}}dy$$

代入得到

$$\int \frac{xdx}{\sqrt{1+\sqrt[3]{x^2}}} = \int \frac{(y^2-1)^{\frac{2}{3}} \cdot 3y(y^2-1)^{\frac{1}{2}}dy}{y} = 3\int (y^2-1)^2dy$$
$$= \int (y^4-2y^2+1)dy = \frac{3}{5}y^5 - 2y^3 + 3y + C$$
$$= \frac{3}{5}(\sqrt{1+\sqrt[3]{x^2}})^5 - 2(\sqrt{1+\sqrt[3]{x^2}})^3 + 3\sqrt{1+\sqrt[3]{x^2}} + C$$

思考题（3）

$(i) \int \frac{dx}{\sqrt{x^2-a^2}}$

ans : 令 $x = a\cosh t$, 则

$$dx = a\sinh tdt, \sqrt{x^2-a^2} = a\sinh t$$

于是得到

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sinh tdt}{a\sinh t} = \int dt = t + C$$
$$= arcosh\frac{x}{a} + C$$

$(ii) \int \frac{dx}{\sqrt{x^2+a^2}}$

ans : 令 $x = a\sinh t$, 则

$$dx = a\cosh tdt, \sqrt{x^2+a^2} = a\cosh t$$

于是得到

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a\cosh t dt}{a\cosh t} = \int dt = t + C$$
$$= arcsinh\frac{x}{a} + C = \ln(\frac{x}{a} + \sqrt{(\frac{x}{a})^2+1}) + C$$
$$= \ln(x + \sqrt{x^2+a^2}) + C^{'}$$

1.3.5. 求积分 $\int \frac{dx}{\sqrt{(x-a)(b-x)}}(a < x < b)$

ans : 设 $x = a\cos^2\varphi + b\sin^2\varphi$, 则

$$dx = -2a\cos\varphi\sin\varphi d\varphi + 2b\cos\varphi\sin\varphi d\varphi = 2(b-a)\cos\varphi\sin\varphi d\varphi$$
$$x - a = a\cos^2\varphi + b\sin^2\varphi - a(\cos^2\varphi + \sin^2\varphi) = (b-a)\sin^2\varphi$$
$$b - x = b(\cos^2\varphi + \sin^2\varphi) - a\cos^2\varphi + b\sin^2\varphi = (b-a)\cos^2\varphi$$

代入得

$$\int \frac{dx}{\sqrt{(x-a)(b-x)}} = \int \frac{2(b-a)\cos\varphi\sin\varphi d\varphi}{\sqrt{(b-a)^2\sin^2\varphi\cos^2\varphi}} = 2\int d\varphi = 2\varphi + C$$
$$= 2\arctan\sqrt{\frac{x-a}{b-x}} + C$$

$$\frac{x-a}{b-x} = \frac{(b-a)\sin^2\varphi}{(b-a)\cos^2\varphi} = \tan^2\varphi$$

所以

$$\tan\varphi = \frac{x-a}{b-x}, \varphi = \arctan\sqrt{\frac{x-a}{b-x}}$$

1.3.6. 计算积分 $\int \frac{dx}{1+\sin x+\cos x}$

ans : 令 $t = \tan\frac{x}{2}, \sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}, dx = \frac{2}{1+t^2}dt$ 代入积分式子

$$\int \frac{dx}{1+\sin x+\cos x} = \int \frac{\frac{2}{1+t^2}dt}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}} = \int \frac{2dt}{2+2t} = \int \frac{dt}{1+t}$$
$$\ln|1+t| + C = \ln|1+\tan\frac{x}{2}| + C$$

1.3.7. 求积分 $\int \frac{dx}{\sqrt{ax^2+bx+c}}$

由 (i) 得到

$$\begin{aligned}\int\frac{dx}{\sqrt{ax^2+bx+c}}&=\int\frac{2\frac{\sqrt{a}\cdot t^2+bt+c\sqrt{a}}{(2\sqrt{a}\cdot t+b)^2}dt}{\frac{\sqrt{a}\cdot t^2+bt+c\sqrt{a}}{2\sqrt{a}\cdot t+b}}=\int\frac{2dt}{2\sqrt{a}\cdot t+b}\\&=\frac{1}{\sqrt{a}}\int\frac{d(2\sqrt{a}\cdot t+b)}{2\sqrt{a}\cdot t+b}\\&=\frac{1}{\sqrt{a}}\ln 2\sqrt{a}\cdot t+b+C\\&=\frac{1}{\sqrt{a}}\ln[2\sqrt{a}(\sqrt{ax^2+bx+c}+\sqrt{a}x)+b]+C\\&=\frac{1}{\sqrt{a}}\ln[2\sqrt{a(ax^2+bx+c)}+2ax+b]+C\end{aligned}$$

1.3.8. 求积分 $\int\frac{dx}{x+\sqrt{x^2-x+1}}$

由 (ii) 得到

$$\begin{aligned}\int\frac{dx}{x+\sqrt{x^2-x+1}}&=\int\frac{-2\frac{t^2-t+1}{(t^2-1)^2}dt}{\frac{t}{t-1}}\\&=-2\int\frac{(t^2-t+1)(t-1)}{(t^2-1)^2t}dt=\int\frac{-2t^2+2t-2}{(t^2-1)^2(t-1)t}dt\\&=\int[\frac{2}{t}-\frac{1}{2}\frac{1}{t-1}-\frac{3}{2}\frac{1}{t+1}-\frac{3}{(t+1)^2}]dt\\&=\frac{3}{t+1}+2\ln|t|-\frac{1}{2}\ln|t-1|-\frac{3}{2}\ln|t+1|+C\end{aligned}$$

代入 $t=\frac{\sqrt{x^2-x+1}+1}{x}$

$$\begin{aligned}\int\frac{dx}{x+\sqrt{x^2-x+1}}=&\frac{3x}{\sqrt{x^2-x+1}+x+1}+2\ln|\frac{\sqrt{x^2-x+1}+1}{x}|\\&-\frac{1}{2}\ln|\frac{\sqrt{x^2-x+1}-x+1}{x}|-\frac{3}{2}\ln|\frac{\sqrt{x^2-x+1}+x+1}{x}|+C\end{aligned}$$

1.3.9. 求积分 $\int\frac{dx}{(x^2+a^2)\sqrt{a^2-x^2}}$

由 (iii) 得到

$$\begin{aligned}
\int \frac{dx}{(x^2+a^2)\sqrt{a^2-x^2}} &= \int \frac{\frac{4at}{(t^2+1)}^2 dt}{\frac{2a^2(t^4+1)}{(t^2+1)^2} \cdot \frac{2at}{t^2+1}} \\
&= \frac{1}{2a^2} \int \frac{2t^2+2}{t^4+1} dt \\
&= \frac{1}{2a^2} \int (\frac{1}{t^2+\sqrt{2}\cdot t+1} + \frac{1}{t^2-\sqrt{2}\cdot t+1}) dt \\
&= \frac{1}{\sqrt{2}a^2} \int \frac{d(\sqrt{2}\cdot t+1)}{(\sqrt{2}\cdot t+1)^2+1} + \frac{1}{\sqrt{2}a^2} \int \frac{d(\sqrt{2}\cdot t-1)}{(\sqrt{2}\cdot t-1)^2+1} \\
&= \frac{1}{\sqrt{2}a^2} \arctan(\sqrt{2}\cdot t+1) + \frac{1}{\sqrt{2}a^2} \arctan(\sqrt{2}\cdot t-1) + C \\
&= \frac{1}{\sqrt{2}a^2} [\arctan(\sqrt{\frac{2(a+x)}{a-x}}+1) + \arctan(\sqrt{\frac{2(a+x)}{a-x}}-1)] + C
\end{aligned}$$